On the generation of surface waves by turbulent shear flows

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(Received 24 March 1959, and in revised form 4 August 1959)

The model proposed by Phillips (1957) for the generation of water waves by the random fluctuations of normal pressure already present in a turbulent wind is generalized to include energy transfer associated with the interaction between surface wave and mean air flow (Miles 1957). It is found that this energy transfer may increase by an order of magnitude the surface displacements produced by a given distribution of pressure fluctuations in the principal stage of development.

1. Introduction

Two models for the generation of surface waves on a nearly inviscid liquid by turbulent winds have been developed recently by Phillips (1957) and Miles (1957). Both models are based on linearized equations of motion, both neglect direct interactions between surface waves and turbulent fluctuations, and both predict total energy-transfer to gravity waves in order-of-magnitude agreement with observation; however, their mechanisms are largely complementary.

The model proposed by Phillips assumes as its essential mechanism the direct action of turbulent fluctuations in aerodynamic pressure on the surface of the water, but neglects all interaction between air flow and surface wave. It is an *uncoupled* model in the sense that excitation is assumed to be independent of response. (See also Eckart (1953) for a similar but rather more artificial model.)

The model proposed by Miles neglects turbulent fluctuations (except as they enter indirectly through the prescription of the mean wind profile) and assumes as its essential mechanism the interaction between mean air flow and surface wave. It is a *coupled* model in the sense that excitation is assumed to be proportional to response, and it finds its antecedants in the classical problems of hydrodynamic stability (and also in the sheltering theory of Jeffreys—see Lamb 1945, §348).

There can be little doubt that both of the foregoing mechanisms are operative in some degree, with that of Phillips providing a broad-spectrum energy-input and that of Miles a frequency-selective feedback. (But we must bear in mind that the two mechanisms are independent only in our simplified model; in reality, the turbulent fluctuations must be affected by the wave motion. It also may be pertinent to observe that, according to Townsend (1956, §6.1), the structure of the largest eddies is determined by the stability of the mean profile.) Assuming linearity, such a feedback may be expected to augment the wave-growth initiated by the pressure fluctuations and lead to an exponential time-dependence, rather than the linear time-dependence (for the power spectral density) that characterizes the original model proposed by Phillips. Non-linear effects, such as a reduction of the energy transfer from the mean flow (cf. Stuart 1958) to the longer waves and breaking of the shorter waves (Phillips 1958), would eventually alter this time-dependence, to be sure, but we remark that the amplitudes of the longest (also largest and fastest) waves are likely to be limited simply by the length of time that the wind blows.

We shall develop here a model that combines these two mechanisms and then seek to determine their relative importance. We consider first (in § 2) the response $\eta(x,t)$ of a free surface to a pressure distribution $p_0(x,t)+p_1(x,t)$, where p_0 denotes a prescribed distribution and p_1 is linearly proportional to η_t; in the context of the preceding discussion, p_1 represents the aerodynamic pressure coupled with the wave motion, but we also may regard it as comprising other effects, such as viscous damping in the water. Eventually (in § 3), we shall assume p_1 to be a stationary random function of (vector position) $\mathbf{x}$ and t and generalize to two-dimensional surface waves, but initially we shall find it expedient to consider the nearly-periodic function

$$p_0(x,t) = f(t)\, e^{ik(x-Vt)}, \qquad |\dot{f}/kVf| \ll 1, \tag{1.1a, b}$$

where $f(t)$ is a slowly varying function of time in the sense of (1.1 b) and V denotes the convection speed. Following Phillips, we then shall focus attention on those (resonant) surface waves that have wave speeds $c = c(k)$ approximating the convection speed V and that may be expressed in the form

$$\eta(x,t) = a(t)\, e^{ik(x-ct)}, \qquad |\dot{a}/kca| \ll 1, \tag{1.2a, b}$$

where $a(t)$ also is a slowly varying function of time. (Assuming η in this form allows c to remain real, whereas in Miles (1957) it had to be assigned a small imaginary part.) After expressing $a(t)$ in terms of $f(t)$, we may extend the analysis to a two-dimensional disturbance, travelling at an angle α with respect to the convected pressure p_0, simply by writing†

$$V = \mathscr{U}(k)\cos\alpha, \tag{1.3}$$

where $\mathscr{U}(k)$ is the convection speed for the pressure fluctuation of wave-number k ($\mathscr{U} = \mathscr{U}_c$ in Phillips's notation). We then (in § 3) may generalize our result to a stationary random p_0 with the aid of Fourier–Stieltjes integrals. These could have been introduced from the beginning, but carrying out the initial analysis in terms of nearly-periodic functions perhaps serves to bring out more simply the essential features of the *resonance mechanism* (discovered by Phillips) and the associated approximations.‡

Phillips considered two stages in the development of surface waves under random pressure: (a) an 'initial stage', during which the spectrum of the pressure fluctuations may be assumed to be independent of time, and (b) a 'principal stage' during which the relatively short waves that influence the wind profile may be

† This is essentially Squire's (1933) result. Alternatively, we may note that if

$$\eta = a(t) \exp\left[ik(x\cos\alpha + y\sin\alpha - ct)\right],$$

the phase velocity in the direction of the convected pressure is simply $c \sec \alpha$.

‡ I am indebted to the referee for this suggestion.

assumed to have achieved a statistical equilibrium and during which only the growth of the longer gravity waves need be considered. We shall give primary attention (in §4) to the principal stage, for it is only then that the interaction between mean flow and *gravity* waves could have developed sufficiently to be important; however, in order to substantiate this last statement, we shall cite (without derivation) the appropriate generalization of Phillips's result for the angular spectrum during the initial stage.

Having combined the two mechanisms of energy transfer in §§2 to 4, we shall develop in §5 an expression for the energy transfer from the mean flow in terms of a Reynolds stress, following a suggestion by Lin (private communication). We then shall show that the energy transfer from the mean flow may be of dominant importance for a significant portion of the gravity-wave spectrum.

2. Resonant surface waves excited by convected pressure

Let $\eta(x,t)$ be the displacement of the free surface of a liquid of density ρ_w that is subjected to a pressure $p(x,t)$, and let p exhibit the x-dependence e^{ikx}. The equation of motion for small disturbances then may be put in the form

$$(\rho_w/k)\,\eta_{tt}+(\rho_w g+Tk^2)\,\eta = -p, \tag{2.1}$$

where ρ_w/k is the effective mass per unit area for an irrotational motion of the liquid, and g denotes the acceleration of gravity and T the surface tension. We do *not* use the moving reference frame adopted by Phillips, but otherwise (2.1) is equivalent to P(2.11). (Here and subsequently, equations from Phillips (1957) will be denoted by P( ), where ( ) contains the equation number.) If $p = 0$, (2.1) has free-surface-wave solutions of the form (1.2a) with $a(t) = \text{const.}$ and

$$c^2 = gk^{-1}+(T/\rho_w)\,k. \tag{2.2}$$

Now let us assume

$$p(x,t) = p_0(x,t)+p_1(x,t), \qquad p_1 = -\zeta\rho_w c\eta_t, \tag{2.3a,b}$$

where p_0 is prescribed by (1.1a,b) and ζ is a small but otherwise arbitrary constant. Substituting (1.1a) and (2.3a,b) in (2.1) and dividing through by ρ_w/k, we then obtain

$$\eta_{tt}-\zeta kc\eta_t+k^2c^2\eta = -(k/\rho_w)f(t)\,e^{ik(x-Vt)}. \tag{2.4}$$

We observe that the complementary solutions to (2.4) are of the form (1.2a) with $a^2(t) = e^{\zeta kct}$, and ζ may be interpreted as the fractional increase in mean energy per radian-cycle according to

$$\zeta = (kc\bar{E})^{-1}(\partial\bar{E}/\partial t), \tag{2.5}$$

where $\bar{E}$ denotes the mean (over an integral number of cycles or wavelengths) energy of the surface wave; $|\zeta| \ll 1$ then implies that $\bar{E}$ must be a slowly varying function of time in the sense of (1.1b) and (1.2b).

We may express the solution of (2.4), subject to the initial conditions $\eta = \eta_t = 0$ at $t = 0$, as

$$\eta(x,t) = \frac{e^{ik(x-Vt)}}{2i\rho_w c}\int_0^t e^{m(t-\tau)}\,[e^{ik(V-c)(t-\tau)}-e^{ik(V+c)(t-\tau)}]f(\tau)\,d\tau, \tag{2.6}$$

where

$$m = \tfrac{1}{2}\zeta kc \ll 1. \tag{2.7}$$

We now invoke our hypotheses that $f(t)$ is a slowly varying function of time and that $V \doteqdot c$; then, after several cycles ($kct \gg 1$), the contribution of the term in $V-c$ will be much more important than that in $V+c$,† and we obtain the *resonance approximation*

$$\eta(x,t) \doteqdot \frac{e^{ik(x-Vt)}}{2i\rho_w c}\int_0^t \exp\{[m+ik(V-c)](t-\tau)\}f(\tau)\,d\tau, \tag{2.8a}$$

if

$$V \doteqdot c. \tag{2.8b}$$

Comparing (2.8a) to (1.2a), we obtain

$$a(t) = \frac{e^{mt}}{2i\rho_w c}\int_0^t \exp\{-[m+ik(V-c)]\tau\}f(\tau)\,d\tau. \tag{2.9}$$

We observe that, subject to the restrictions (1.1b), (2.7) and (2.8b), $a(t)$ must be a slowly varying function of time, as anticipated in (1.2b).

3. Response of free surface to random pressure

Following Phillips, we now suppose the pressure acting on the free surface to have the stochastic form

$$p_0(\mathbf{x},t) = \int \exp[i\mathbf{k}.(\mathbf{x}-\mathfrak{U}t)]\,d\varpi(\mathbf{k},t), \tag{3.1}$$

where $\mathbf{x}$ denotes vector position on the free surface, $\mathbf{k}$ is a (two-dimensional) vector wave-number of magnitude k (Phillips used κ where we use k), $\mathfrak{U}(k)$ the convection velocity of that differential component of p_0 having the wave-number k, and the integral is of the Fourier–Stieltjes type. We also define α as the angle between $\mathfrak{U}$ (which is assumed to have a fixed direction, namely, that of the wind) and the wave-front normal of a given component of $\eta(\mathbf{x},t)$ (see (3.2) below); we then may regard k and α as the polar (spectral) co-ordinates of $\mathbf{k}$. These definitions correspond to those adopted by Phillips, but we again emphasize that we have not used his moving reference frame, so that where he writes $\mathbf{x}$ we write $\mathbf{x}-\mathfrak{U}t$; this avoids any possible difficulties associated with the fact that, in general, the convection speed $\mathscr{U}$ depends on k.

Corresponding to (3.1), we may assume the free-surface response to have the form

$$\eta(\mathbf{x},t) = \int \exp[i\mathbf{k}.(\mathbf{x}-\mathfrak{U}t)]\,dA(\mathbf{k},t). \tag{3.2}$$

Comparing (3.1) and (3.2) with (1.1a) and (1.2a) and regarding $d\varpi(\mathbf{k},t)$ as the counterpart of $f(t)$, the corresponding counterpart of $a(t)$ is

$$\exp[i(kc-\mathbf{k}.\mathfrak{U})t]\,dA(\mathbf{k},t).$$

Identifying V with $\mathscr{U}\cos\alpha$, as in (1.3), we then may generalize the resonance approximation (2.8a) according to

$$dA(\mathbf{k},t) = \frac{1}{2i\rho_w c}\int_0^t \exp\{[m+ik(V-c)](t-\tau)\}\,d\varpi(\mathbf{k},\tau)\,d\tau \tag{3.3a}$$

if

$$V = \mathscr{U}(k)\cos\alpha \doteqdot c(k). \tag{3.3b}$$

† The argument here is essentially similar to that invoked in Kelvin's stationary-phase approximation. Alternatively, taking the Laplace transform of (2.6) and allowing the transform parameter to tend to zero (corresponding to $kct \gg 1$) reveals that the respective integrals are proportional to $[m^2+k^2(V-c)^2]^{-\frac{1}{2}}$ and $[m^2+k^2(V+c)^2]^{-\frac{1}{2}}$.

Still following Phillips, we now introduce the amplitude spectrum

$$\Phi(\mathbf{k},t) = \frac{\overline{dA(\mathbf{k},t)\,dA^*(\mathbf{k},t)}}{dk_1\,dk_2}, \tag{3.4}$$

where the asterisk denotes the complex conjugate, the bar implies an ensemble (or equivalent) average, and $dk_1\,dk_2\,(=k\,dk\,d\alpha)$ is the element of area in the $\mathbf{k}$-plane. Similarly, we introduce the pressure spectrum

$$\Pi(\mathbf{k},t) = \frac{\overline{d\varpi(\mathbf{k},t')\,d\varpi^*(\mathbf{k},t+t')}}{dk_1\,dk_2}, \tag{3.5}$$

which is a function of the time separation t. Substituting (3.3*a*) in (3.4) then yields

$$\Phi(\mathbf{k},t) = \frac{1}{4\rho_w^2c^2}\int_0^t\int_0^t \exp\left[m(2t-\tau-\tau')+ik(V-c)(\tau'-\tau)\right] \times\frac{\overline{d\varpi(\mathbf{k},\tau)\,d\varpi^*(\mathbf{k},\tau')}}{dk_1\,dk_2}\,d\tau\,d\tau' \tag{3.6a}$$

$$= \frac{1}{4\rho_w^2c^2}\int_0^t\int_0^t \exp\left[m(2t-\tau-\tau')+ik(V-c)(\tau'-\tau)\right]\Pi(\mathbf{k},\tau-\tau')\,d\tau\,d\tau', \tag{3.6b}$$

where (3.6*b*) follows from (3.6*a*) in accordance with (3.5). Introducing $\tau+\tau'$ and $\tau-\tau'$ as new variables of integration, noting that $\Pi(\mathbf{k},t)$ is an even function of t, carrying out the $\tau+\tau'$ integration, and then replacing the dummy variable $\tau-\tau'$ by τ, we obtain

$$\Phi(\mathbf{k},t) = \frac{e^{mt}}{2\rho_w^2c^2m}\int_0^t \sinh\left[m(t-\tau)\right]\cos\left[k(V-c)\,\tau\right]\Pi(\mathbf{k},\tau)\,d\tau. \tag{3.7}$$

We emphasize that the result (3.7) is valid only over that portion of the k-spectrum for which (3.3*b*) may be considered a satisfactory approximation.

4. The principal stage; gravity waves

We consider now what Phillips has termed the principal stage of development, in which t is assumed to exceed the development time of $d\varpi(\mathbf{k},t)$. This implies that the relatively short waves that influence the mean velocity profile have achieved (a statistical) equilibrium and that only the growth of the longer gravity waves need be considered.

We consider first the asymptotic approximation of the integral in (3.7). Let

$$g_1(t) = \int_0^t g_2(\tau)\sinh\left[m(t-\tau)\right]d\tau, \qquad g_2(\tau) = \Pi(\mathbf{k},\tau)\cos\left[k(V-c)\,\tau\right], \tag{4.1a, b}$$

and let capitals denote Laplace transforms according to

$$G(s) = \int_0^\infty e^{-st}g(t)\,dt. \tag{4.2}$$

Transforming (4.1*a*) with the aid of the convolution theorem, we obtain

$$G_1(s) = m(s^2-m^2)^{-1}\,G_2(s). \tag{4.3}$$

We then may obtain the (first term in the) asymptotic approximation to $g(t)$ for $t \gg T$, where T denotes the time scale of $g_2(t)$, simply by setting $s = 0$ in $G_2(s)$; on the other hand, we have *not* assumed $mt \gg 1$ and therefore do not approximate $s^2 - m^2$ by $-m^2$. It follows that

$$g_1(t) \sim G_2(0)\sinh(mt) = \sinh(mt)\int_0^\infty g_2(\tau)\,d\tau, \tag{4.4}$$

whence we obtain from (3.7)

$$\Phi(\mathbf{k}, t) \sim \frac{F(t,m)}{2\rho_w^2 c^2}\int_0^\infty \Pi(\mathbf{k},\tau)\cos[k(V-c)\tau]\,d\tau \tag{4.5a}$$

$$= (2\rho_w^2 c^2)^{-1} F(t,m)\,\Pi(\mathbf{k},0)\,\theta(\mathbf{k}, c\sec\alpha - \mathscr{U}), \tag{4.5b}$$

where
$$F(t,m) = (2m)^{-1}(e^{2mt} - 1), \tag{4.6}$$

and θ denotes the integral time-scale of the pressure fluctuations (as defined by Phillips).

Expanding (4.6) according to

$$F(t,m) = t + mt^2 + \ldots \quad (mt \to 0), \tag{4.7}$$

we observe that (4.5a) reduces to P(4.3) as $mt \to 0$ provided that the resonance approximation (which implies neglecting $\cos[k(V+c)\tau]$ relative to $\cos[k(V-c)\tau]$ in the integral) is invoked in P(4.3) and a minor error therein is corrected by multiplying it by $\sqrt{2}$.†

We conclude from equations (4.5) to (4.7) that the relative importance of energy transfer through the direct action of the pressure fluctuations and energy transfer through the interaction between surface wave and mean flow depends (within the assumptions implicit in our model) only on the size of mt relative to unity. In particular, the shape of Φ *vs* k is independent of t if $mt \ll 1$, but otherwise it is not.

We also note that if Π is assumed to be independent of $\tau - \tau'$ in (3.7), as in the initial stage considered by Phillips, and (following Phillips) the angular spectrum $\Psi(\alpha)$ is defined according to

$$\overline{\eta^2} = \iint \Phi k\,dk\,d\alpha = \int_0^{\alpha_{cr}} \Psi(\alpha)\,d\alpha, \tag{4.8}$$

where
$$\alpha_{cr} = \cos^{-1}(c_{\min}/\mathscr{U}) = \cos^{-1}(4gT/\rho\mathscr{U}^4)^{\frac{1}{4}}, \tag{4.9}$$

then the time-dependent factor in Ψ, which was simply t in Phillips's result, also becomes $F(t,m)$ for the present model.

5. Energy transfer from the mean flow

We turn now to the calculation of ζ, as defined by (2.5). Assuming that the motion is two-dimensional and inviscid, it is known that the rate at which energy is transferred from the mean flow to the disturbance is given by (Lin 1955, equations (4.5.1, 2))

$$\frac{\partial E}{\partial t} = -\rho\iint (\overline{uv})\,\overline{u}_y\,dx\,dy, \tag{5.1}$$

† Phillips also has found that P(4.3) *et seq.* for Φ require the correction factor $\sqrt{2}$ (private communication).

where u and v denote the x- and y-components of velocity and ρ the density. Assuming the flow to consist of a small disturbance with respect to the parallel shear flow $U(y)$, we may take $\bar{u} = U(y)$. We also assume the small disturbance to exhibit the x-dependence e^{ikx}, as in (1.2a), and take mean values over x. The Reynolds stress in (5.1) then may be evaluated according to (Lin 1954)†

$$-\rho_a \overline{uv} = -\rho_a(\pi U_c''/kU_c')\overline{v_c^2} \quad y < y_c, \tag{5.2a}$$

$$= 0 \quad y > y_c, \tag{5.2b}$$

where the subscript c implies evaluation at the critical layer defined by

$$U(y_c) = c \quad (\alpha = 0), \tag{5.3a}$$

or, anticipating the extension to obliquely moving waves,

$$U(y_c)\cos\alpha = c. \tag{5.3b}$$

Substituting (5.2a, b) in (5.1) and taking the mean value over x, we obtain

$$\left(\overline{\frac{\partial E}{\partial t}}\right) = -\rho_a \int_0^{y_c} (\overline{uv}) U'(y)\, dy = -\rho_a c(\pi U_c''/kU_c')\overline{v_c^2}. \tag{5.4}$$

The mean kinetic and potential energies for the surface wave of (1.2) are equal if $a(t)$ is constant and approximately equal if $a(t)$ is slowly varying in the sense of (1.2b); accordingly, the mean total energy is given by

$$\bar{E} = \rho_w k^{-1}\overline{\eta_t^2}. \tag{5.5}$$

Substituting (5.4) and (5.5) in (2.5), we obtain

$$\zeta = -s(\pi U_c''/kU_c')(\overline{v_c^2}/\overline{\eta_t^2}), \tag{5.6}$$

where
$$s = \rho_a/\rho_w. \tag{5.7}$$

The calculation of $\overline{v_c^2}/\overline{\eta_t^2}$, the ratio of the mean-square velocity at the inner critical layer to the mean-square velocity of the surface wave, requires the solution of the inviscid Orr–Sommerfeld equation. Only a rough, integral approximation was given originally by Miles (1957), but the equation has since been integrated numerically (Conte & Miles 1959) for the logarithmic profile

$$U(y) = U_1 \log(y/z_0), \qquad U_1 = U_*/\kappa, \tag{5.8a, b}$$

where U_* denotes Prandtl's shearing-stress velocity, κ ($\doteqdot$ 0·4) Karman's constant, and z_0 the effective roughness parameter. Both the approximate and direct (numerical) integrations of the differential equation lead to the conclusion that if the parameter β is introduced according to

$$\zeta = s(U_1\cos\alpha/c)^2\beta, \tag{5.9}$$

where U_1/c is replaced by $U_1\cos\alpha/c$ to allow for oblique waves, then

$$\beta \doteqdot \beta(ky_c) \quad (ky_0 < ky_c < 2), \tag{5.10}$$

† The neglect of viscosity in calculating the Reynolds stress can be justified (as an approximation) only if $U_c''/U_c' < 0$, corresponding to positive energy-transfer; otherwise, the inviscid solution may not be significant for mathematically small but physically finite values of the viscosity.

where y_0 denotes the lower limit of validity for the logarithmic profile. Within the indicated range for ky_c, β does not depend on the parameter kz_0,† which enters only indirectly through the relation between ky_c and $c/U_1 \cos\alpha$; for gravity waves this relation may be placed in the form (from (5.3*b*) and (5.8*a*))

$$ky_c = \Omega(U_1 \cos\alpha/c)^2 \exp(c/U_1 \cos\alpha), \tag{5.11a}$$

where

$$\Omega = gz_0/U_1^2 \cos^2\alpha. \tag{5.11b}$$

The numerical results for β are plotted in figures 1 and 2, with Ω appearing as the family parameter in the latter. We emphasize that, since β is proportional to the profile curvature $U''(y_c)$, the assumption of a logarithmic profile may lead to appreciable error even though the approximation to $U(y)$ is close; this is especially so for the lower portion of the profile.

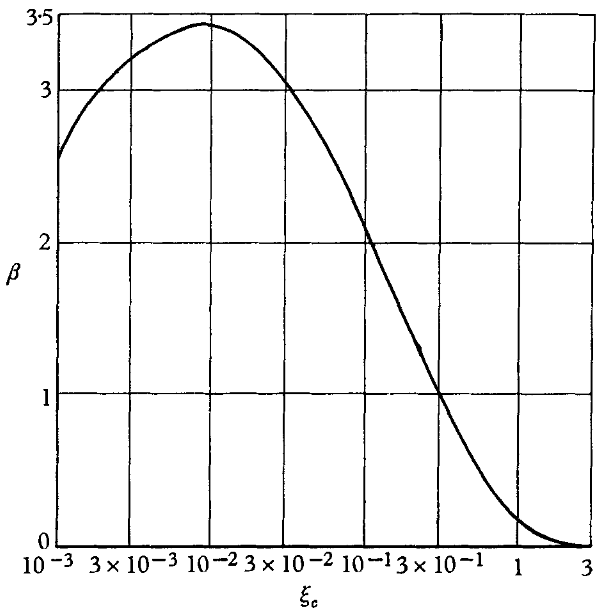

FIGURE 1. β *vs* ky_c; see (5.9).

Returning now to the results of § 4, we shall consider possible values of $F(t, m)$, as defined in (4.5) and (4.6). Letting

$$k = g/c^2 \tag{5.12}$$

on the assumption of gravity waves and evaluating ζ from (5.9), we obtain

$$2mt = \zeta kct = s\beta(U_1 \cos\alpha/c)^2 (gt/c). \tag{5.13}$$

Referring to the data cited from Sverdrup & Munk by Phillips in his figure 6, and assuming that the convection speed $\mathscr{U}$ for gravity waves can be approximated by the anemometer speed U and that the latter is given approximately by

$$U = 9U_1, \tag{5.14}$$

† The referee has suggested that, similarly, $\Pi = \rho_a^2 U_1^4 k^{-2} P(ky_c, \alpha)$, where the dependence of P on α is much less 'dramatic' than its dependence on ky_c. This suggestion might simplify the evaluation of the integrals in (3.7) and (4.5), but it still remains to determine P.

as for wind speeds of the order of 10 m/sec at an elevation (of the anemometer) of 10 m and $\Omega \doteqdot 10^{-2}$, we present some representative calculations in table 1. We also add that calculations carried out for the initial stage lead to the conclusion $mt \ll 1$ for all t and k for which the pressure spectrum may be considered independent of t and for which energy transfer from the mean flow exceeds that dissipated by viscosity in the water (which may be included by adding the negative increment $-2\nu_w k^2$ to m, where ν_w is the kinematic viscosity of water (Stokes 1850)).

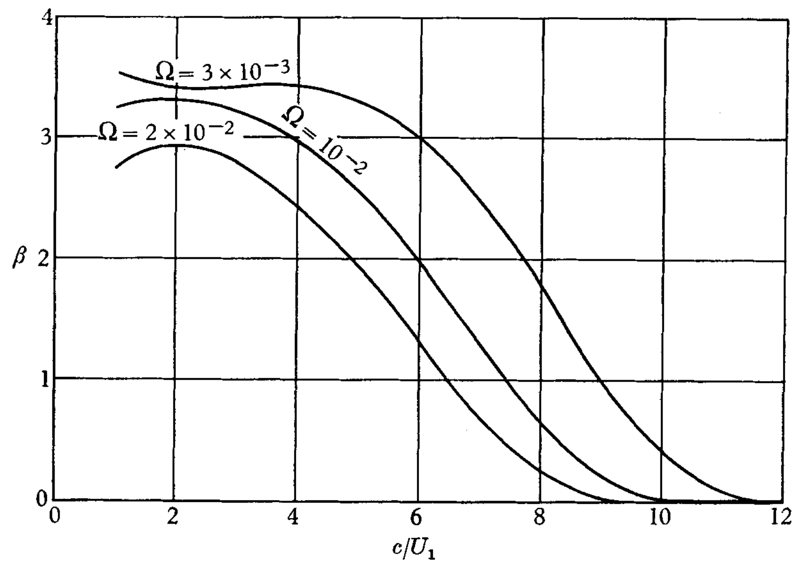

FIGURE 2. β *vs* c/U_1; $\Omega = gz_0/U_1^2$; U_1 must be replaced by $U_1 \cos\alpha$ if $\alpha \neq 0$.

gt/U	c/U	α	β	ζkct	$t^{-1}F(t, m)$
10^5	1	0°	0·19	0·28	1·14
10^5	0·5	0°	2·8	33	10^{13}
10^5	0·5	45°	1·1	6·5	10^2
10^4	0·5	0°	2·8	3·3	8

TABLE 1. Representative calculations based on (5.13) and (4.6) with $\rho_a/\rho_w = 1{\cdot}2 \times 10^{-3}$ and $\Omega = 10^{-2}\sec^2\alpha$. The values of β have been obtained from figure 2.

It seems obvious from the values of $F(t, m)$ indicated in table 1 that the inclusion of energy transfer from the mean flow yields a total energy-transfer to the surface waves that is likely to be much greater than that predicted (in its absence) by P(4.13); on the other hand, we recall that the rate of energy transfer from the mean flow to a *prescribed spectrum of surface waves* appears to be in order-of-magnitude agreement with observation (Miles 1959). The fact that P(4.13) gives order-of-magnitude agreement with observation then suggests that at least one of the approximations invoked by Phillips might be appreciably in error. Of these approximations, perhaps the most uncertain are those leading to his estimate of root-mean-square pressure fluctuation as $(\overline{p^2})^{\frac{1}{2}} \doteqdot 0{\cdot}6(\frac{1}{2}\rho_a U^2)$. This is roughly an order of magnitude larger than the values estimated by Eckart (1953), but in the absence of actual measurements the question must be regarded as open.†

It might be objected, to be sure, that the very large values of ζkct indicated in table 1 imply that the assumption of small disturbances cannot remain valid. It is entirely possible, however, that linearized theory does give the right order of magnitude for the energy transfer from the mean flow (even though there might be some non-linear reduction thereof) and that non-linear effects are dominant only in the dissipative mechanism. Such an interpretation is supported by the agreement with observation cited above (Miles 1959).

Another objection that may be lodged against our model (although not for the values of c/U in table 1) is that, for wind speeds of the order of 10 m/sec, there must be a significant portion of the gravity-wave spectrum for which y_c is so small that the logarithmic profile cannot remain valid. It remains true, nevertheless, that in the absence of viscosity some energy will be transferred from the mean flow to the surface waves if $U_c''/U_c' < 0$, as is evident from (5.4); moreover, the inclusion of viscosity (in the air) in the calculation actually can *increase* this energy transfer (see Benjamin 1959; Miles 1959). It also is likely that appreciable energy could be transferred through 'sheltering'—i.e. form drag, involving a flow that alternately separates from and reattaches to the wave crests (Lamb 1945, §348).

6. Conclusions

We conclude that energy transfer associated with interaction between surface waves and mean air flow in the excitation of these waves by prescribed, random fluctuations of normal pressure may increase the total energy-transfer by an order of magnitude, at least under those conditions for which non-linear effects may be neglected. We emphasize, however, that a decisive assessment of our results depends on a more precise determination of the pressure fluctuations than any presently available.

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† Phillips also has concluded that his estimate of $(\overline{p^2})^{\frac{1}{2}}$ was too large (private communication).